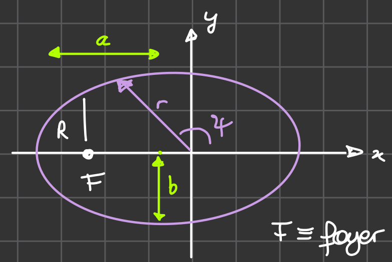
Problem set 2: GWs from eccentric binaries

We will now extend our study of GWs to binary systems with non-zero eccentricity. Indeed, many astrophysical binary system exhibit significant eccentricity, particularly in their early inspiral phases.

We will derive key orbital parameters, computing radiated power and examining how GW backreaction influences orbital evolution

I. ELLIPTIC ORBITS

In a 2-body motion, we have 2 conserved quantities: energy and angular momentum



$$\begin{cases} L = \mu r^2 \dot{\psi} \\ E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\psi}^2\right) - G\frac{\mu m}{r} \end{cases}$$

We also have

$$r(\psi) = \frac{R}{1+e\cos\psi} \quad ; \quad R = \frac{L^2}{Gm\mu^2} \quad ; \quad e^2 = 1 + \frac{2EL^2}{G^2m^2\mu^3},$$

with R a length scale of the system and e the eccentricity: $0 < e < 1$

We can express

$$a = \frac{R}{1-e^2} = \frac{L^2/Gm\mu^2}{-2EL^2/G^2m^2\mu^3} = \frac{Gm\mu}{2|E|}$$

where taking |.| is important because we have a gravitational system with bound orbit

$$E = -\frac{G m_1 m_2}{2a}$$

and we want to ensure that a is positive. We can then express the equation of the orbit as

$$r(\psi) = \frac{a(1-e^2)}{1+e\cos\psi}.$$

We can now compute the mass moments. We find in the Problem Set 1 that for 2 point-like masses,

$$M^{ij} = m x^i_{CM} x^j_{CM} + \mu x^i_0 x^j_0$$

and choosing $x_{CM} = 0$, this reduces to a one-body problem with

$$M^{ij} = \mu x^i_0 x^j_0.$$

In the (x,y)-plane,

$$\begin{cases} x_0(t) = r(t)\cos\psi(t) \\ y_0(t) = r(t)\sin\psi(t) \end{cases}$$

st

$$M^{ij} = \mu r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix}.$$

If we want to compute the radiated power, in the quadrupolar approximation, we have

$$P_{quad.} \sim \langle \dddot{M}\,\dddot{M} \rangle.$$

For example, let's take

$$\begin{aligned} M_{11} &= \mu r^2 \cos^2\psi \\ &= \mu a^2 (1-e^2)^2 \frac{\cos^2\psi}{(1+e\cos\psi)^2} \\ &= C_1 \cdot \frac{\cos^2\psi}{(1+e\cdot\cos\psi)^2} \end{aligned} \qquad C_1 = \mu a^2 (1-e^2)^2$$

To compute its time derivative, we need to know $\dot\psi$.

$$\begin{aligned} \dot\psi = \frac{L}{\mu r^2} = \frac{(GmR)^{1/2}}{r^2} &= \frac{\sqrt{GmR}}{a^2(1-e^2)^2}(1+e\cos\psi)^2 \qquad R = a(1-e^2) \\ &= \left(\frac{Gm}{a^3}\right)^{1/2} \frac{(1+e\cos\psi)^2}{(1-e^2)^{3/2}} \\ &= C_2 \cdot (1+e\cos\psi)^2 \end{aligned} \qquad C_2 = \left(\frac{Gm}{a^3}\right)^{1/2} \frac{1}{(1-e^2)^{3/2}}$$

This allows us to compute

- $$\begin{aligned} \dot M_{11} &= C_1 \cdot \frac{-2\cos\psi\sin\psi\,\dot\psi\,(1+e\cos\psi)^2 + 2\cos^2\psi\,(1+e\cos\psi)\,e\sin\psi\,\dot\psi}{(1+e\cos\psi)^4} \\ &= C_1 \frac{-2\cos\psi\sin\psi\,\dot\psi - \cancel{2e\cos^2\psi\sin\psi\,\dot\psi} + \cancel{2\cos^2\psi\, e\sin\psi\,\dot\psi}}{(1+e\cos\psi)^3} \\ &= -C_1 \frac{\sin(2\psi)}{(1+e\cos\psi)^3}\dot\psi \\ &= -C_1 C_2 \frac{\sin(2\psi)}{1+e\cos\psi} \end{aligned}$$

- $$\begin{aligned} \ddot M_{11} &= -C_1 C_2 \frac{2\cos(2\psi)\dot\psi(1+e\cos\psi) + \sin(2\psi)\,e\sin\psi\,\dot\psi}{(1+e\cos\psi)^2} \\ &= -C_1 C_2^2 \left(2\cos(2\psi) + 2e\cos^3\psi\right) \end{aligned}$$

$\cos(2\psi) = 2\cos^2\psi - 1$

$2\cos(2\psi)\cos\psi + \sin(2\psi)\sin\psi$
$= \cos(2\psi)\cos\psi + \cos\psi$
$= 2\cos^3\psi$

- $$\begin{aligned} \dddot M_{11} &= -2C_1 C_2^2 \left(-2\sin(2\psi)\dot\psi - 3\cos^2\psi\sin\psi\,\dot\psi\, e\right) \\ &= \beta(1+e\cos\psi)^2\left(2\sin(2\psi) + 3e\sin\psi\cos^2\psi\right) \end{aligned}$$

with $\beta = \dfrac{2G^{3/2}\mu m^{3/2}}{a^{5/2}(1-e^2)^{5/2}} = C_1 C_2^3.$

We do the same for the remaining components and one find

$$\begin{cases} \dddot{M}_{22} = \beta (1+e\cos\psi)^2 \left[-2\sin(2\psi) - e\sin\psi\ (1+3\cos^2\psi) \right] \\ \dddot{M}_{12} = \beta(1+e\cos\psi)^2 \left[-2\cos(2\psi) + e\cos\psi\,(1-3\cos^2\psi) \right] \end{cases}$$

Going back to computing the radiated power in the quadrupolar approximation, we have

$$\begin{aligned} P_{quad} &= \frac{G}{5c^5} \langle \dddot{Q}_{ij}\, \dddot{Q}_{ij} \rangle \\ &= \frac{G}{5c^5} \langle \dddot{M}_{ij}\,\dddot{M}_{ij} - \frac{1}{3} (\dddot{M}_{kk})^2 \rangle \\ &= \frac{G}{5c^5} \langle \dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 2\dddot{M}_{12}\,\dddot{M}_{12} - \frac{1}{3}(\dddot{M}_{11} + \dddot{M}_{22})^2 \rangle \\ &= \frac{8G^4}{15c^5} \cdot \frac{\mu^2 m^3}{a^5(1-e^2)^5} \langle (1+e\cos\psi)^4 \left(12(1+e\cos\psi)^2 + e^2\sin^2\psi \right) \rangle \end{aligned}$$

We then need to compute this average wrt time.⊗ Because of the periodicity of the considered motion, we only focus on one orbital period:

⊗ GW energy taken instantaneously fluctuates widely over short timescales, making it difficult to extract meaningful physics
→ average power smooths out these fluctuations
see 1.4.3 of Maggiore for more insights

$$\begin{aligned} P_{quad} &= \frac{1}{T} \int_0^T dt \,.\, P(\psi(t)) \\ &= \frac{1}{T} \int_0^{2\pi} d\psi \,.\, (\dot{\psi})^{-1}\, P(\psi) \end{aligned}$$

and because we have a Keplerian motion,

$$\frac{1}{T} = \frac{\omega_0}{2\pi} = \left(\frac{Gm}{a^3} \right)^{1/2} \frac{1}{2\pi}$$

$$\begin{aligned} \Rightarrow P_{quad} &= \frac{8G^4\mu^2 m^3}{15c^5 a^5} (1-e^2)^{-7/2} \frac{1}{2\pi} \int d\psi \left[12(1+\cos\psi)^4 + e^2(1+e\cos\psi)^2 \sin^2\psi \right] \\ &= \frac{8G^4\mu^2 m^3}{15c^5 a^5} f(e) \end{aligned}$$

with $f(e) = \dfrac{1}{(1-e^2)^{7/2}} \left(1 + \dfrac{73}{24} e^2 + \dfrac{37}{96} e^4 \right)$.

Note that if we have a circular trajectory, $f(0) = 1$ and $a = R$ → we recover the result of Problem Set 1.

Effects due to the eccentricity are not negligeable. Moreover, this formula is historically meaningful. Let us look at the orbital period

$$\begin{aligned} T = \frac{2\pi}{\omega_0} &= 2\pi\, a^{3/2} (Gm)^{-1/2} \\ &= 2\pi\,(Gm)^{-1/2} \left(\frac{Gm\mu}{2|E|} \right)^{3/2} \\ &= C\,.\,|E|^{-3/2} \end{aligned}$$

$$\Rightarrow \frac{\dot{T}}{T} = \frac{(|E|^{-3/2})^{\cdot}}{|E|^{-3/2}} = \frac{3}{2} \cdot \frac{\dot{E}}{|E|} = -\frac{3}{2} \frac{32 G^4 \mu^2 m^3}{15 c^5 a^5} f(e) \frac{2a}{Gm\mu}$$

$(\dot{E} = -P)$

$$= -\frac{96}{5} \left(\frac{T}{2\pi}\right)^{-8/3} \frac{(Gm)^{-4/3} G^3 m^2 \mu}{c^5} f(e)$$

$$= -\frac{96}{5} \left(\frac{T}{2\pi}\right)^{-8/3} \frac{G^{5/3} m^{2/3} \mu}{c^5} f(e)$$

$\rightarrow$ result from 1993

Hulse-Taylor pulsar : PSR B1913+16 (NS + other star)

Pulsar : NS turning relatively fast and that keeps an quite important magnetic field. While turning, it projects a flux of particles.

Its semi-major axis reduces of 3.5 m / year

II. BACKREACTION ON ELLIPTIC ORBITS

You have seen that

$$\frac{dL_i}{dt} = -\frac{2G}{5c^5} \varepsilon^{ikl} \langle \ddot{Q}_{ka} \dddot{Q}_{la} \rangle$$

see 3.3.4 Maggiore

$$= -\frac{2G}{5c^5} \varepsilon_{ikl} \langle \ddot{M}_{ka} \dddot{M}_{la} \rangle$$

due to sym - antisym because

$$Q^{ij} = M^{ij} - \frac{1}{3} \delta^{ij} M_{kk}$$

and

$$\varepsilon_{ikl} \delta^{ka} \dddot{Q}^{la} = \underbrace{\varepsilon_{ikl}}_{\text{antisym}} \underbrace{\dddot{Q}^{kl}}_{\text{sym}}$$

Choosing $L \equiv L_z$, one obtains

$$\frac{dL}{dt} = -\frac{2G}{5c^5} \langle \ddot{M}^{1a} \dddot{M}^{2a} - \ddot{M}^{1a} \dddot{M}^{2a} \rangle$$

$$= \frac{4G}{5c^5} \langle \dot{M}_{12} (\ddot{M}_{11} - \ddot{M}_{22}) \rangle$$

$\equiv 2.5$PN

Let's compute these derivatives:

$$M_{12} = \mu r^2 \sin\psi \cos\psi = \frac{1}{2} \mu a^2 (1-e^2)^2 \frac{\sin(2\psi)}{(1+e\cos\psi)^2} = C_1 \frac{\sin(2\psi)}{(1+e\cos\psi)^2}$$

$$\Rightarrow \begin{cases} \dot{M}_{12} = 2 C_1 C_2 \dfrac{e\cos\psi + \cos(2\psi)}{1+e\cos\psi} \\ \ddot{M}_{12} = -\dfrac{\mu G m}{a(1-e^2)} \cdot \left[4\cos\psi + e\,(3+\cos(2\psi))\right] \sin\psi \end{cases}$$

$$\Rightarrow \frac{dL}{dt} = -\frac{32}{5} \cdot \frac{G^{7/2} \mu^2 m^{5/2}}{c^5 a^{7/2}} \frac{1}{(1-e^2)^2} \left(1 + \frac{7}{8} e^2\right)$$

(averaged on one period)

Remember that we already found

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G^4\mu^2 m^3}{c^5 a^5}\cdot\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right).$$

We are gonna re-express this diff. eq. system in terms of a and e. Remember that

$$\begin{cases} e^2 = 1 + \dfrac{2EL^2}{G^2m^2\mu^3} \\ a = \dfrac{R}{1-e^2} = \dfrac{Gm\mu}{2|E|} \end{cases}$$

$$\Leftrightarrow \begin{cases} E = -\dfrac{Gm\mu}{2a} \\ L = \sqrt{(1-e^2)\,Gm\mu^2 a} \end{cases}$$

and

- $$\frac{dE}{dt} = \frac{dE}{da}\cdot\frac{da}{dt} = \frac{Gm\mu}{2a^2}\cdot\frac{da}{dt}$$

$$\Rightarrow \frac{da}{dt} = -\frac{64}{5}\cdot\frac{G^3\mu m^2}{c^5a^3}\cdot\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

↳ semi-major axis decreases while the system emits GWs

- $$\frac{dL}{dt} = \frac{\partial L}{\partial a}\cdot\frac{da}{dt} + \frac{\partial L}{\partial e}\cdot\frac{de}{dt}$$

$$= \frac{1}{2}\sqrt{\frac{(1-e^2)\,Gm\mu^2}{a}}\cdot\frac{da}{dt} - \sqrt{\frac{e^2\,Gm\mu^2 a}{(1-e^2)}}\cdot\frac{de}{dt}$$

$$\Rightarrow \frac{de}{dt} = \frac{1}{e}\sqrt{\frac{1-e^2}{Gm\mu^2 a}}\left[-\frac{32}{5}\cdot\frac{G^{7/2}\mu^2m^{5/2}}{c^5a^{7/2}}\cdot\frac{1}{(1-e^2)^3}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\right.$$
$$\left.+\frac{32}{5}\cdot\frac{G^{7/2}\mu^2m^{5/2}}{c^5a^{7/2}}\cdot\frac{(1-e^2)}{(1-e^2)^3}\left(1+\frac{7}{8}e^2\right)\right]$$

$$= -\frac{304}{15}\cdot\frac{G^3\mu m^2}{c^5a^4}\cdot\frac{1}{(1-e^2)^{5/2}}\,e\left(1+\frac{121}{304}e^2\right)$$

↳ GWs circularised orbits: $\begin{cases} e=0, \text{ trajectory stays circular } \left(\frac{de}{dt}=0\right) \\ e>0, \ \frac{de}{dt}<0, \text{ trajectory becomes less eccentric} \end{cases}$

Solve those eqs numerically is not an easy task. We have to adimensionalise the time

$$\tau = \frac{ct}{R_*}$$ ≡ time measured in units of the light travel time across a distance R_* (≡ Schw)

$\sim 10^{12}$ ...

An analytical way to solve this problem is to obtain $a(e)$:

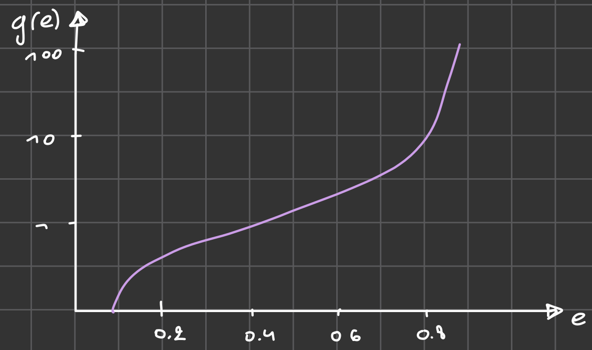
$$\frac{da}{de} = \frac{da}{dt}\left(\frac{de}{dt}\right)^{-1}$$

$$= \frac{12}{19}\cdot\frac{a}{e(1-e^2)}\cdot\frac{1+73/24\, e^2 + 37/96\, e^4}{1+121/304\, e^2} \quad \rightarrow \text{integrable}$$

$$\vdots$$

$$\Rightarrow a(e) = C_0'\,\frac{e^{12/19}}{1-e^2}\left(1+\frac{121}{304}e^2\right)^{\frac{870}{2299}}$$

$$= C_0\, g(e)$$



Going back to our favourite pulsar, we know that today, it has

$$\begin{cases} a_0 \simeq 2.10^9\,\text{m} \\ e_0 \simeq 0{,}617 \end{cases}$$

By the time the two stars reach a short separation a, say $a \simeq \mathcal{O}(10^2 R_{NS}) = 10^3\,\text{km}$, we have

$$\frac{a}{a_0} = \mathcal{O}(5\ 10^{-4})$$

and since $g(e_0) = \mathcal{O}(1)$,

$$\frac{a(e)}{a(0)} = \frac{g(e)}{g(0)} \sim 10^{-3} \Rightarrow \quad e \sim (5.10^{-4})^{19/12} \sim 6.10^{-6}$$

$\Rightarrow$ Unless some external interaction perturbs the system, long before the 2 NS approach the coalescence phase, the ellipticity $\simeq 0$ $\rightarrow$ circular orbit

When we observe GWs, we are near the coalescence.